CATENARY AROUND A (2+1)-DIMENSIONAL STATIC BLACK HOLE

Hiromi SUZUKI*

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Abstract

I obtained exact solutions to the string equations derived from the Polyakov action in a (2+1)-dimensional black hole under a special anzatz in the last year's Science Reports [1]. In this paper, I show exact string solutions and catenaries around a (2+1) -dimensional black hole without putting any special anzatz.

§ 1. Exact String Solution around a (2+1)-Dimensional Black Hole

The black hole solution [2–4] with a black hole mass M is given by

$$ds^2=\left(M-\frac{r^2}{l^2}\right)dt^2+\left(\frac{r^2}{l^2}-M\right)^{-1}dr^2+r^2d\varphi^2 . \tag{1.1}$$

Generally, the string equations of motion [5–12] are

$$\partial^2 X^A+\Gamma^A_{BC}(X)\partial_+X^B\partial_-X^C=0 , \tag{1.2}$$

where Γ^A_{BC} are the Christoffel connections associated to the space metric, $x_1=1/2(\tau\pm\sigma)$ with τ and σ being the string world sheet coordinates. The string constraints on the world sheet are written as

$$T_{\pm\pm}=G_{AB}(X)\partial_\pm X^A\partial_\pm X^B=0 . \tag{1.3}$$

The string equations and constraints can be derived from the Polyakov action:

$$S_{\rm sch}=\frac{1}{2\alpha'}\int d^2x\sqrt{-g}g^{ab}G_{\mu\nu}(x)\partial_a X^\mu\partial_b X^\nu , \tag{1.4}$$

by applying the variational principle to the string coordinates X^A and the world sheet metric g_{ab}. Then we have chosen a gauge:

$$g_{+-}=1 , \qquad g_{++}=g_{--}=0 , \tag{1.5}$$

by exploiting the two dimensional diffcomorphism and the conformal invariance. In the case of BTZ black hole, the string equations of motion are

* E-mail: hsuzuki@th.phys.titech.ac.jp

$$\partial_+\partial_-\hat{t}+\frac{\hat{r}}{l^2}\left(\frac{\hat{r}^2}{l^2}-1\right)^{-1}(\partial_+\hat{t}\partial_-\hat{r}+\partial_+\hat{r}\partial_-\hat{t})=0\,, \tag{1.6}$$

$$\partial_+\partial_-\hat{r}+\frac{\hat{r}}{l^2}\left(\frac{\hat{r}^2}{l^2}-1\right)(\partial_+\hat{t}\partial_-\hat{t}-l^2\partial_+\hat{\varphi}\partial_-\hat{\varphi})$$
$$-\frac{\hat{r}}{l^2}\left(\frac{\hat{r}^2}{l^2}-1\right)^{-1}\partial_+\hat{r}\partial_-\hat{r}=0\,, \tag{1.7}$$

$$\partial_+\partial_-\hat{\varphi}+\hat{r}^{-1}(\partial_+\hat{r}\partial_-\hat{\varphi}+\partial_+\hat{\varphi}\partial_-\hat{r})=0\,, \tag{1.8}$$

and the constraint equations are

$$-\left(\frac{\hat{r}^2}{l^2}-1\right)\partial_+\hat{t}\partial_+\hat{t}+\left(\frac{\hat{r}^2}{l^2}-1\right)^{-1}\partial_+\hat{r}\partial_+\hat{r}+\hat{r}^2\partial_+\hat{\varphi}\partial_+\hat{\varphi}=0\,, \tag{1.9}$$

$$-\left(\frac{\hat{r}^2}{l^2}-1\right)\partial_-\hat{t}\partial_-\hat{t}+\left(\frac{\hat{r}^2}{l^2}-1\right)^{-1}\partial_-\hat{r}\partial_-\hat{r}+\hat{r}^2\partial_-\hat{\varphi}\partial-\hat{\varphi}=0\,. \tag{1.10}$$

Here, let $\hat{r}/l=\mathrm{ch}\,\theta$, and change the coordinates $(\hat{t},\hat{\varphi})$ to $(l\tau,\lambda)$, using $\partial_\pm=2(\partial_\tau\pm\partial_\sigma)$, equations (1.6)–(1.8) become

$$\theta=\theta(\sigma) \tag{1.11}$$

$$-\partial_\sigma^2\theta(\sigma)+\mathrm{ch}\,\theta\,\mathrm{sh}\,\theta(1-(\partial_\tau\lambda+\partial_\sigma\lambda)(\partial_\tau\lambda-\partial_\sigma\lambda))=0 \tag{1.12}$$

$$(\partial_\tau+\partial_\sigma)(\partial_\tau-\partial_\sigma)\lambda-2\,\mathrm{th}\,\theta\cdot\partial_\sigma\lambda\partial_\sigma\theta=0 \tag{1.13}$$

$$-\mathrm{sh}^2\theta+(\partial_\sigma\theta)^2+\mathrm{ch}^2\theta(\partial_\tau\lambda+\partial_\sigma\lambda)^2=0 \tag{1.14}$$

$$-\mathrm{sh}^2\,\theta+(\partial_\sigma\theta)^2+\mathrm{ch}^2\,\theta(\partial_\tau\lambda-\partial_\sigma\lambda)^2=0\,. \tag{1.15}$$

It follows that equations (1.14) and (1.15) hold only in the case of $\lambda=\lambda(\sigma)$ or $\lambda=\lambda(\tau)$. Here, only a solution for $\lambda=\lambda(\sigma)$ will be shown for simplicity. Inserting equations (1.12)–(1.14) into $\lambda=\lambda(\sigma)$, equations (1.12)–(1.14) become

$$-\partial_\sigma^2\theta+\mathrm{sh}\,\theta\,\mathrm{ch}\,\theta(1+(\partial_\sigma\lambda)^2)=0 \tag{1.16}$$

$$\partial_\sigma^2\lambda+2\,\mathrm{th}\,\theta\cdot\partial_\sigma\lambda\partial_\sigma\theta=0 \tag{1.17}$$

$$-\mathrm{sh}^2\theta+(\partial_\sigma\theta)^2+\mathrm{ch}^2\,\theta(\partial_\sigma\lambda)^2=0\,. \tag{1.18}$$

From equation (1.17),

$$\partial_\sigma\lambda\cdot\mathrm{ch}^2\,\theta=A\,,\qquad (A\text{: constant}) \tag{1.19}$$

is obtained. Inserting equation (1.19) into equation (1.18),

$$\hat{r}=l\sqrt{1+\frac{(\sqrt{1+4A^2}-1)/2}{\mathrm{cn}^2\left(\sqrt[4]{1+4A^2}\sigma,\ \dfrac{\sqrt{(1+\sqrt{1+4A^2})/2}}{\sqrt[4]{1+4A^2}}\right)}} \tag{1.20}$$

is obtained. This solution is an exact string solution around a (2+1)-dimensional static black hole, which is obtained without putting any anzatz.

§ 2. Catenary in a (2+1)-Dimensional Black Hole

Equation (3.19) is an exact string solution with two end points (Fig. 1). As shown in Fig. 1, R_0 denotes the distance between the center of a black hole and one of the end points and α, the angle between the one of the end points and the point of the string nearest to the black hole. To summarise, we have

$$-\sigma_1 \leqq \sigma \leqq \sigma_1 \tag{2.1}$$

$$\begin{cases} l\,\mathrm{ch}\,\theta(\sigma=0)=l\sqrt{\dfrac{1+\sqrt{1+4A^2}}{2}}\equiv L & (2.2) \\ l\,\mathrm{ch}\,\theta(\sigma=\pm\sigma_1)=R_0 & (2.3) \end{cases}$$

$$\begin{cases} \lambda(\sigma=0)=0 & (2.4) \\ \lambda(\sigma=\sigma_1)=\alpha\,. & (2.5) \end{cases}$$

From equations (1.20) and (2.3),

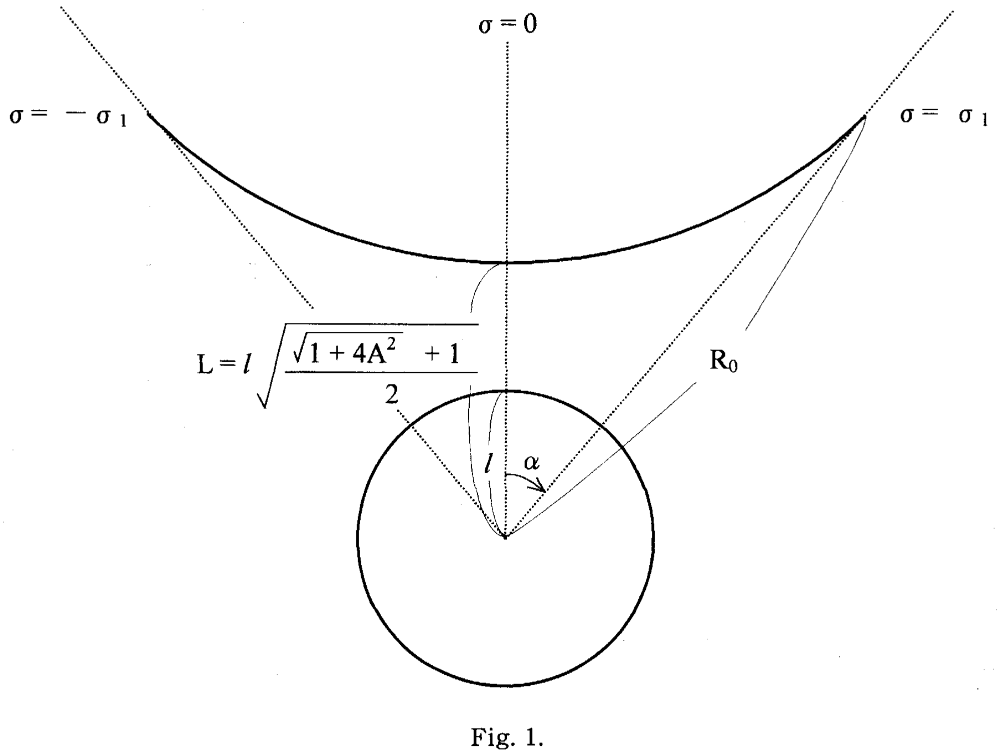


Fig. 1.

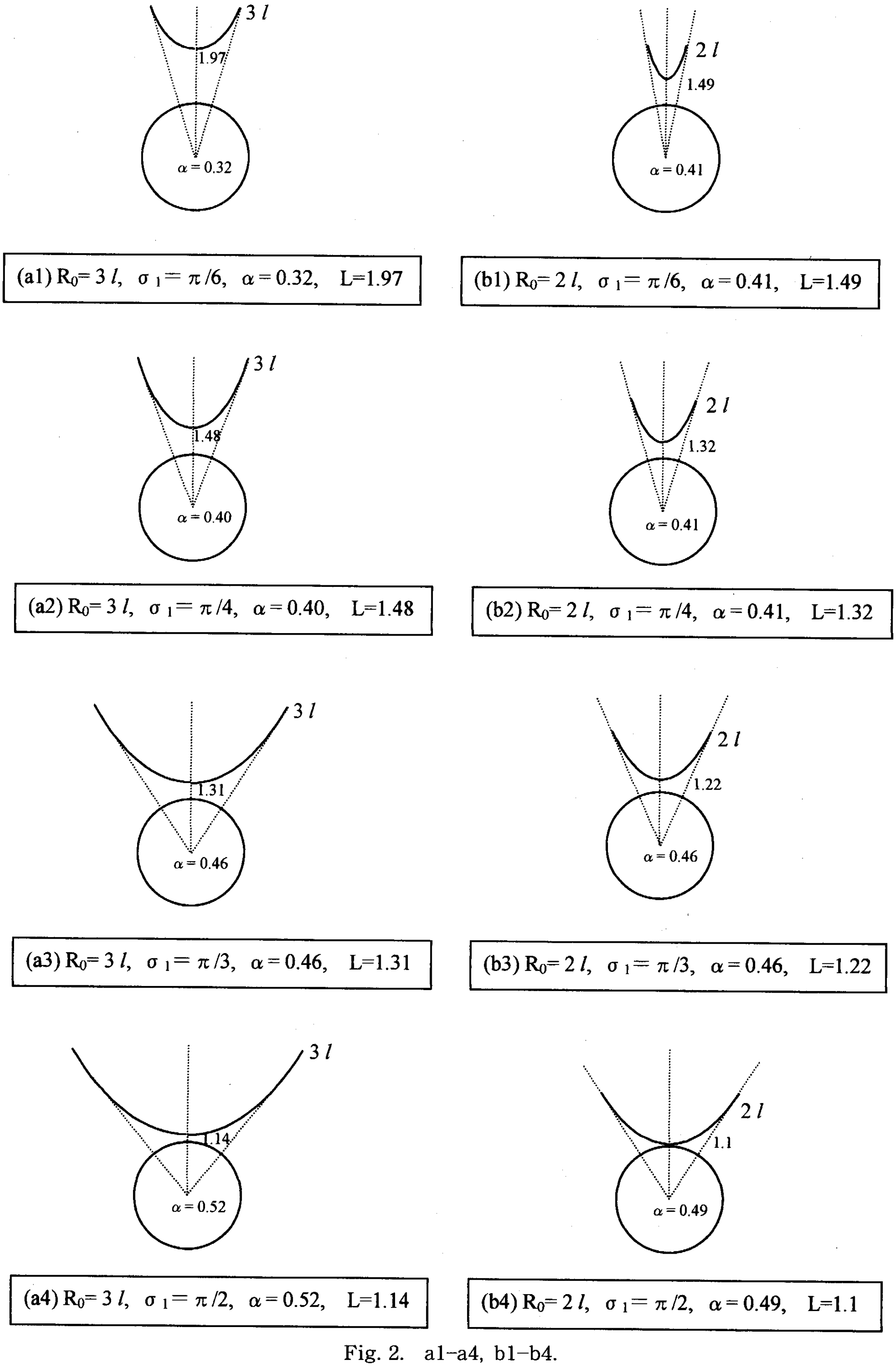


Fig. 2. a1–a4, b1–b4.

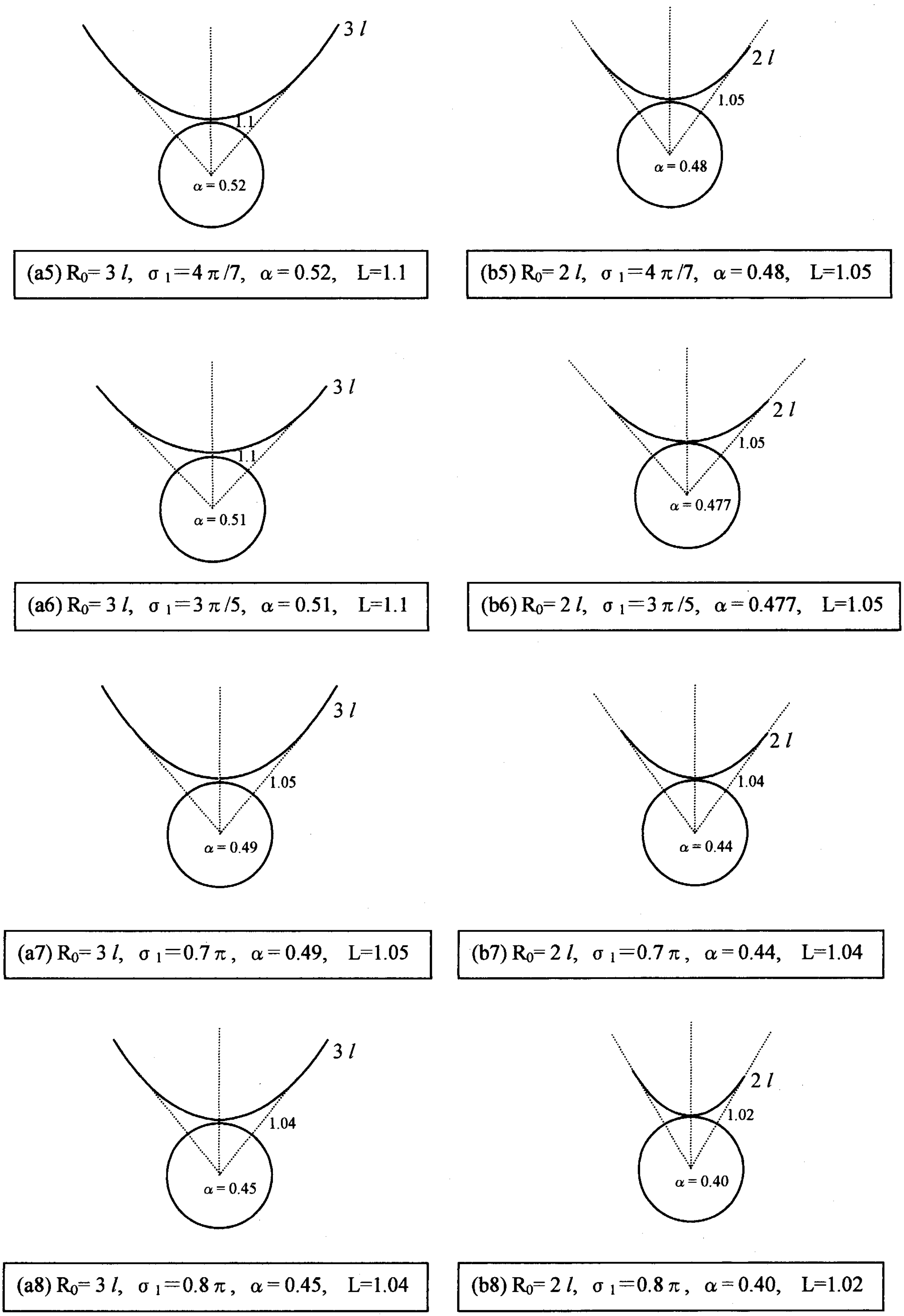
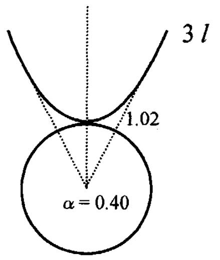
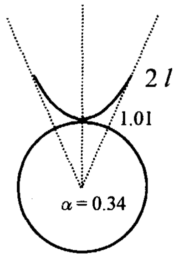


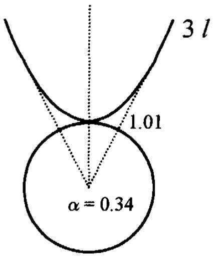
Fig. 2. a5–a8, b5–b8.



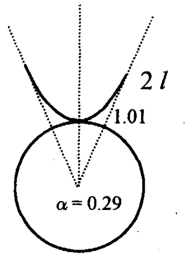
(a9) $R_0 = 3\,l$, $\sigma_1 = 0.9\pi$, $\alpha = 0.40$, L=1.02



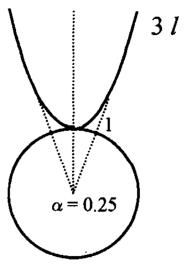
(b9) $R_0 = 2\,l$, $\sigma_1 = 0.9\pi$, $\alpha = 0.34$, L=1.01



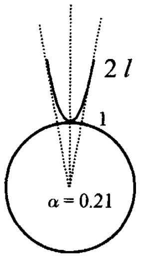
(a10) $R_0 = 3\,l$, $\sigma_1 = \pi$, $\alpha = 0.34$, L=1.01



(b10) $R_0 = 2\,l$, $\sigma_1 = \pi$, $\alpha = 0.29$, L=1.01



(a11) $R_0 = 3\,l$, $\sigma_1 = 7\pi/6$, $\alpha = 0.25$, L=1



(b11) $R_0 = 2\,l$, $\sigma_1 = 7\pi/6$, $\alpha = 0.21$, L=1

Fig. 2. a9–a11, b9–b11.

$$\frac{\sqrt{1+4A^2}-1}{\left(2\,\mathrm{cn}^2\left(\sqrt[4]{1+4A^2}\sigma_1,\ \dfrac{\sqrt{(1+\sqrt{1+4A^2})/2}}{\sqrt[4]{1+4A^2}}\right)\right)}=\left(\frac{R_0}{l}\right)^2-1 \qquad (2.6)$$

is derived. Using equations (1.19) and (2.3), α is given as

$$\alpha=\int_0^{\sigma_1}\frac{A\,\mathrm{cn}^2\left(\sigma\sqrt[4]{1+4A^2},\ \dfrac{\sqrt{(1+\sqrt{1+4A^2})/2}}{\sqrt[4]{1+4A^2}}\right)}{\mathrm{cn}^2\left(\sigma\sqrt[4]{1+4A^2},\ \dfrac{\sqrt{(1+\sqrt{1+4A^2})/2}}{\sqrt[4]{1+4A^2}}\right)+\dfrac{\sqrt{1+4A^2}-1}{2}}\,d\sigma. \qquad (2.7)$$

Once R_0 and α are given by the equations (2.6) and (2.7), the two equations including A and σ_1 are obtained. This means that A and σ_1 are explicitly given (Fig. 2). This offers an equation for a catenary with the two end points fixed in a (2+1)-dimensional black hole. The catenaries with $R_0=2$ and 3 are depicted in Fig. 2 when σ_1 varies from 0 to $(7/6)\pi$. In the case of $0 \leq \sigma_1 \leq (\pi/2)$, the catenary gradually approaches a black hole with the increase in σ_1, the value of α also being increasing. In the case of $\sigma_1 > (\pi/2)$, the catenary a little bit comes close to a black hole gradually with the increase in σ_1, the value of α being decreasing. The suitable explanation for the behavior of α with the increase in σ_1 in the case of $\sigma_1 > (\pi/2)$ still remains an open question to me.

§ 3. Conclusion and Discussion

A catenary solution for the string equation around a (2+1)-dimensional static black hole is exactly obtained.

The catenaries with $R_0=2$ and 3 are illustrated in Fig. 2 when σ_1 varies from 0 to $(7/6)\pi$. The reason for the behavior of α in the case of $\sigma_1 > (\pi/2)$ still remains to be explored.

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